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WEIGHT FUNCTIONS FOR EXTERNAL CIRCULAR CRACKS

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Abstract—In this paper the three-dimensional (3D) weight functions are derived for external circular cracks. The solution method used is similar to that developed by Bueckner (*Int. J. Solids Structures* **23**, 57–93 (1987)) for internal circular cracks lying in infinite elastic solids. A Papkovitch–Neuber potential is used to represent the tensile mode weight function field. This potential, derived from some known solutions to a mixed boundary value potential problem by Galin (*Contact Problems in the Theory of Elasticity*. School of Physical Sciences and Applied Mathematics, North Carolina State College (1953)), also uniquely determines the shear mode weight function fields. The results for internal circular cracks by Bueckner are presented for comparison and for completeness. For external circular cracks, different forms of the weight functions exist corresponding to different displacement boundary conditions at infinity. The Neuber fields, denoting the elastic fields of an external circular crack due to remote forces and/or moments, are used to determine the weight functions under various boundary conditions. The crack face weight functions, defined as the intensity factors induced by a pair of equal, oppositely sensed unit point forces acting on the upper and lower crack faces, are presented in closed formulae. In the Appendices the present results are checked against some existing solutions, e.g. intensity factor solutions due to the point forces acting along the central axis normal to the crack plane.

INTRODUCTION

The concept of "weight functions" was first introduced by Bueckner (1970) for two-dimensional (2D) elastic crack analysis. In Bueckner's work, the weight functions constitute the displacement field of a special elastic field which he referred to as a "fundamental field". A fundamental field satisfies the Navier displacement equations, equilibrates zero body forces and surface tractions. The displacements of that field are of inverse square root singularity in distance from a crack tip, in contrast to the normal square root dependence of regular displacement fields. Applying Betti's theorem of reciprocity to the fundamental field and the regular elastic field of a crack, Bueckner showed that the weighted average of applied forces with the weight functions gives the crack tip stress intensity factors. This gives a primitive interpretation of the weight functions as the point force solutions for the stress intensity factors. Shortly after Bueckner's work, Rice (1972) developed his weight function concepts in a different way, showing that 2D weight functions could be determined by differentiating the elastic displacement field with respect to crack length; hence the knowledge of a 2D elastic crack solution for any one loading allows the crack solution to be determined for the same body under any other loading systems. Following these works there has been a vast literature on application of weight functions on 2D crack analysis.

The three-dimensional (3D) theory of weight functions, extending Bueckner's 2D concepts, was developed independently by Rice (1972), based on displacement field variations associated to first order with an arbitrary variation in position of the crack front, and by Bueckner (1973), based on a 3D analog of fundamental fields that equilibrate null forces with arbitrary distributions of strength of a normally inadmissible singularity along the crack front. The 3D weight functions not only give stress intensity factors along a crack front for arbitrary body force and surface force distributions, but also determine the first-order variation in the displacement field associated with an arbitrary change in crack front position (Rice, 1985a). The latter property further allows the complete elastic field of a cracked body to be determined by integration over a crack size variable from an uncracked state just before the introduction of the crack, to the actual cracked state.

Rice (1985a) further developed a linear perturbation approach that determines the first-order variation of the elastic field for a crack being slightly perturbed from some simple

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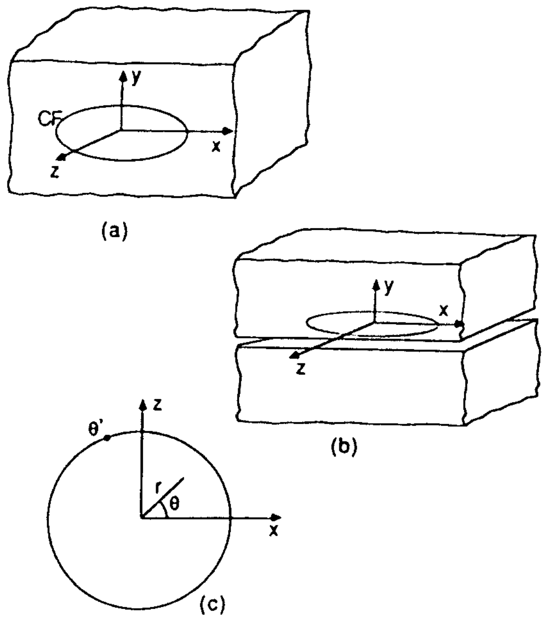

Fig. 1. (a) An internal and (b) external circular crack in an infinite elastic body. (c) The polar coordinates r, θ in the crack plane; the arc length location θ' along the crack front.

reference shape. The perturbation results can be used to address the configurational stability of crack front shapes during quasi-static crack growth and also to model the crack growth in the medium of locally non-uniform fracture toughness. Moreover, Rice (1985b) showed that the weight function concept can be used to describe the 3D crack interactions with sources of internal stresses such as transformation strains and dislocations. These powerful properties of 3D weight functions motivate the search for their explicit solutions for various crack geometries.

Circular cracks are among those most heavily studied crack problems in fracture mechanics, partly because of their mathematical simplicity. Bueckner (1977) determined the tensile mode weight function field for an internal "penny-shaped" circular crack, and recently he (Bueckner, 1987) completed the derivation also for the shear modes by solving for a class of fundamental fields represented by a Papkovitch–Neuber potential function. In the following we extend Bueckner's method to solve for the weight functions for an external circular crack, formed as two elastic half spaces joined over a circular connection area. The solutions for internal circular cracks are presented for comparison and for completeness.

TENSILE MODE WEIGHT FUNCTION FIELDS

An isotropic, homogeneous elastic body is considered that is cracked inside or outside a circular area with radius a, corresponding to an internal (Fig. 1(a)) or an external circular crack (Fig. 1(b)). A fixed Cartesian coordinate system x, y, z is attached to the circular crack system so that the crack lines on the plane $y = 0$ and the origin of the coordinate system is assumed to coincide with the center of cracks (more precisely, it is the center of the circular connection for an external circular crack). Also for circular geometry one sets up associated polar coordinates r, θ in the x–z plane, with θ being zero along the positive x-axis and increases toward the positive z-axis. Weight functions are denoted as $h_{\alpha j}(r, \theta, y\,;\theta'\,;a)$, which can be interpreted as the mode α intensity factor induced at location θ' along the crack front by a unit point force in the j ($j = x, y, z$ or r, θ, y) direction at position r, θ, y. The dependence of $h_{\alpha j}$ on the crack radius a is explicitly emphasized.

One may observe that the elasticity problem of a planar crack lying on the $y = 0$ plane in an infinite elastic body, subjected to a pair of unit wedge opening forces on the crack faces can be formulated by Papkovitch–Neuber potential representations so that the displacement field is expressed as (Galin, 1953)

$$u_x = [(1+\nu)/E](F_{,x}+yY_{,x})$$
$$u_y = -2[(1-\nu^2)/E]Y+[(1+\nu)/E]yY_{,y}$$
$$u_z = [(1+\nu)/E](F_{,z}+yY_{,z}) \tag{1}$$

where F and Y are harmonic functions related by $F_{,y} = (1-2\nu)Y$. Again a comma is used to denote differentiations, e.g. $F_{,y} = \partial F/\partial y$. By symmetry one knows that the displacement u_y should be an odd function of y, and so is the harmonic function Y. The stress components that enter crack surface boundary conditions are calculated from stress–strain relations as

$$\sigma_{yy} = -Y_{,y}+yY_{,yy}, \quad \sigma_{yx} = yY_{,yx}, \quad \sigma_{yz} = yY_{,yz}. \tag{2}$$

It is seen from eqns (2) that there is no shear traction on the $y = 0$ plane. Thus the problem of a wedge opening point force pair at position x', z' (corresponding to r', θ' in polar coordinates) on the crack face is one of finding a function Y satisfying $\nabla^2 Y = 0$, vanishing at infinity, and generating stress $\sigma(x,z) = \delta(x-x')\delta(z-z')$ (Dirac δ functions) within the crack area, and zero opening gap, i.e. $\Delta u(x,z) = 0$ outside the crack area on $y = 0$. Hence by eqns (1) and (2)

$$Y_{,y}|_{y=0} = -\delta(x-x')\delta(z-z'), \quad \text{inside crack area}$$
$$Y|_{y=0^+} = 0, \quad \text{outside crack area.} \tag{3}$$

Note that an external circular crack system may have rigid body displacements at infinity under the action of point forces. In that case the requirement $Y = 0$ at infinity corresponds to imposing certain restraints there to suppress the rigid displacements. The reaction forces and/or moments associated with those displacement restraints contribute to the total stress intensity factors at the crack tip. For more relaxed displacement boundary conditions such as the completely free (no restraints) condition, some or all of those remote reaction forces and/or moments should be taken off by superposing equal and oppositely sensed net forces and/or moments, and their corresponding contributions in the total stress intensity factor expressions should be subtracted off, respectively. In this manner one can obtain different forms of the weight functions, each associated with a different remote displacement boundary condition. Further discussion on this will be left to a later section and for the present it will be assumed that all displacements vanish at infinity.

The solution for the harmonic function Y that satisfies mixed boundary value conditions (3) on a circular region with radius a (i.e. $Y|_{y=0}$, $Y_{,y}|_{y=0}$ given inside and outside a circle) can be extracted from Galin (1953). Under the respective boundary conditions for internal ($Y = Y^i$) and external ($Y = Y^e$) circular cracks, they are expressed in cylindrical coordinates r, θ, y as follows:

$$Y^i(r,\theta,y;r',\theta') = -\frac{\operatorname{sgn}(y)}{\pi^2 d}\arctan\left\{\frac{\sqrt{((a^2-r'^2)(a^2-r^2-y^2+R))}}{\sqrt{2}ad}\right\} \tag{4}$$

$$Y^e(r,\theta,y;r',\theta') = -\frac{\operatorname{sgn}(y)}{\pi^2 d}\arctan\left\{\frac{\sqrt{((r'^2-a^2)(r^2+y^2-a^2+R))}}{\sqrt{2}ad}\right\} \tag{5}$$

where

$$d^2 = r^2-2rr'\cos(\theta-\theta')+r'^2+y^2$$
$$R^2 = (a^2-r^2-y^2)^2+4a^2y^2 \tag{6}$$

and sgn $(y) = y/|y|$. The full elasticity problem for a circular tensile mode crack subjected to a pair of wedge point forces on the crack faces are then solved in terms of the potential functions Y^i and Y^e. By superposition one can further calculate the stresses and displacements under any distribution of applied forces.

It can be observed that by substituting eqn (4) or eqn (5) into eqns (1) one can compute the displacements u_x, u_y and u_z at (r, θ, y) due to a pair of unit opening point forces acting on the crack faces at $(r', \theta', 0^{\pm})$, for the respective internal and external circular crack cases. By the elastic reciprocal theorem, those very same results for u_x, u_y and u_z also represent the opening gaps Δu_y on the crack faces induced at $(r', \theta', 0)$ by unit point forces at (r, θ, y) in the respective x-, y- and z-directions. One knows that in the near vicinity of the front of an external circular crack the opening gap Δu_y is asymptotically related to the mode I stress intensity factor by

$$\Delta u_y \sim \frac{8(1-\nu^2)}{E}\sqrt{\left(\frac{r'-a}{2\pi}\right)}K_1. \tag{7}$$

The same relation holds for an internal circular crack if one replaces $r'-a$ by $a-r'$ in the above equation. Equation (7) is an illustrative example of general asymptotic relations between the stress intensity factors and the crack face displacement discontinuities. Therefore, from the knowledge of the opening gap Δu_y in the vicinity of the crack front, one may also calculate the tensile mode stress intensity factors induced by the unit point forces at (r, θ, y) in the respective x-, y- and z-directions. These stress intensity factors define the x-, y- and z-components of the tensile mode vector weight function h_{1j}. That is

$$h_{1j}(r, \theta, y; \theta'; a) = \lim_{r' \to a} \frac{E}{8(1-\nu^2)}\sqrt{\left(\frac{2\pi}{r'-a}\right)}u_j(r, \theta, y; r', \theta', 0). \tag{8}$$

The reciprocal interpretation of the 3D weight functions in eqn (8) generalizes an interpretation given by Paris *et al.* (1976) in the 2D case.

The weight functions in eqn (8) are expressed proportional to the displacement field induced by a pair of unit point forces on the crack faces, as the force location approaching the crack front. Similar relations can be generalized to the shear modes. The above suggests that the weight functions can be thought of as the displacements of some special elastic field. This field actually corresponds to Bueckner's fundamental field. Equation (8) shows that the vector weight function $h_{\alpha j}$ satisfies the elastic Navier displacement equations.

Now use the cylindrical coordinates formed by r, θ, y. Let a, θ' denote an observation point along the crack front. The limit in eqn (8) leads to the following representations for the tensile mode weight functions h_{1j} $(j = r, \theta, y)$:

$$\begin{aligned} h_{1r} &= -(1-2\nu)G^*_{,r} - yG^*_{,ry}; \quad h_{1y} = 2(1-\nu)G^*_{,y} - yG^*_{,yy}; \\ h_{1\theta} &= -(1-2\nu)G^*_{,\theta}/r - yG^*_{,\theta y}/r \end{aligned} \tag{9}$$

where for internal circular cracks, $G^* = G^i$ is related to the function $Y^i(r, \theta, y; r', \theta')$ by

$$G^i_{,y} = \lim_{r' \to a} \frac{E}{8(1-\nu^2)}\sqrt{\left(\frac{2\pi}{a-r'}\right)}Y^i = \frac{\sqrt{2}\,\mathrm{sgn}\,(y)}{8(1-\nu)\sqrt{(a\pi^3)}}\,\frac{\sqrt{(a^2-r^2-y^2+R)}}{d^2} \tag{10}$$

and for external circular cracks $G^* = G^e$

$$G^{e}_{,y} = \lim_{r \to a} \frac{E}{8(1-v^2)} \sqrt{\left(\frac{2\pi}{r'-a}\right)} Y^{e} = \frac{\sqrt{2}\,\mathrm{sgn}\,(y)}{8(1-v)\sqrt{(a\pi^3)}} \frac{\sqrt{(r^2+y^2-a^2+R)}}{d^2} \tag{11}$$

where now $d^2 = r^2 - 2ar\cos(\theta'-\theta) + a^2 + y^2$. The harmonic functions G^i and G^e are Papkovitch–Neuber potentials representing the special fundamental field of tensile mode weight functions in eqns (9). One can verify that $G^i_{,y} \sim 1/y^2$, while $G^e_{,y} \sim 1/y$ as y approaches infinity so that the function G^e itself may not vanish at infinity. The weight function fields are related to derivatives of the potentials G^i or G^e, and hence one can leave these potentials indefinite within an arbitrary constant. It is obvious that both potentials G^e and G^i are even functions of variable y and one can write

$$G^i = \int^y G^i_{,y}\,\mathrm{d}y, \quad G^e = \int^y G^e_{,y}\,\mathrm{d}y. \tag{12}$$

In order to derive the explicit forms for G^i and G^e from their derivatives with respect to y, follow Bueckner (1987) in adopting oblate spheroidal coordinates s, t which are related to the present cylindrical coordinates by the following:

$$r + \mathrm{i}y = a\cosh(s + \mathrm{i}t) \tag{13}$$

where $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit. The oblate spheroidal coordinates s, t are convenient to describe circular cracks. One lets s, t be defined on the region $0 < s < \infty$, $-\pi/2 < t < \pi/2$ for internal circular cracks and defined on $-\infty < s < \infty$, $0 < t < \pi/2$ for external circular cracks. With these choices the crack faces are represented by $s = 0$ ($t > 0$ on the upper face and $t < 0$ on the lower face) for internal circular cracks, and by $t = 0$ ($s > 0$ on the upper face and $s < 0$ on the lower face) for external circular cracks.

The following relations should be kept in mind:

$$s_{,r} = t_{,y} = \sinh s \cos t/aN; \quad s_{,y} = -t_{,r} = \cosh s \sin t/aN \tag{14}$$

where $N = \sinh^2 s + \sin^2 t$. For conciseness of presentation, from now on complex variables are used during calculations, understanding that the real parts of the final results are implied for various real quantities such as the weight functions. One can further express $G^i_{,y}$ and $G^e_{,y}$ in terms of variables s, t as

$$G^i_{,y} = A\frac{\cosh^2 s + q}{\cosh^2 s - q}\frac{\sin t}{aN} = A\frac{\cosh^2 s + q}{\cosh^2 s - q}\frac{1}{\cosh s}s_{,y}$$

$$G^e_{,y} = A\frac{q + \cos^2 t}{q - \cos^2 t}\frac{\sinh s}{aN} = A\left(\frac{\sinh s}{aN} + \frac{2\cos t}{q - \cos^2 t}t_{,y}\right) \tag{15}$$

where $q = (r/a)\exp \mathrm{i}(\theta - \theta')$ and $A = 1/[4(1-v)\sqrt{(a\pi^3)}]$. Note that q is independent of variable y. Equation $(15)_1$ suggests that $G^i = G^i(s, q)$, as observed by Bueckner (1987), so that by directly carrying out the integration and letting G^i vanish at infinity one obtains the explicit form for G^i as

$$G^i = A\left\{\frac{1}{\sqrt{(q-1)}}\log\frac{\sinh s - \sqrt{(q-1)}}{\sinh s + \sqrt{(q-1)}} + \pi/2 - \arctan(\sinh s)\right\} \tag{16}$$

which agrees with the solution derived by Bueckner (1987). Integration of eqn $(15)_2$ needs more care. Through some analysis one finds that it is not valid to assume $G^e = G^e(t, q)$. Since on the y-axis $t = \pi/2 = \text{const}$ yet one knows G^e has to depend on the variable y on the y-axis, therefore G^e should depend on all three variables s, t, q. However, if one separates G^e into the following:

$$G^e = G_0^e + G_s^e \tag{17}$$

with

$$G_{0,y}^e = A \sinh s/aN, \quad G_{s,y}^e = \frac{2A \cos t}{q - \cos^2 t} t_{,y}. \tag{18}$$

One now observes that $G_{s,y}^e \to 0$ as $r \to 0$, i.e. as the y-axis is approached. Therefore, G_s^e is constant along the y-axis. For simplicity one can set that G_s^e vanishes along the y-axis, i.e. $G_s^e|_{t=\pi/2} = 0$. It is then valid to assume $G_s^e = G_s^e(t, q)$. The s dependence of G^e is absorbed in the axisymmetric part G_0^e. One can then carry out the integrations in eqns (18) to write

$$G_0^e = A \log [\cosh s(1 + \sin t)]$$
$$G_s^e = \frac{A}{\sqrt{(1-q)}} \log \frac{(\sin t - \sqrt{(1-q)})\,(1 + \sqrt{(1-q)})}{(\sin t + \sqrt{(1-q)})\,(1 - \sqrt{(1-q)})}. \tag{19}$$

A constant has been dropped in the integration of G_0^e and the real parts of the right-hand sides are implied in the above eqns (19). The potential function G^e for external circular cracks is given by $G^e = G_0^e + G_s^e$. The complex function $\sqrt{(1-q)}$ is defined on a branch such that $\mathrm{Re}\,[\sqrt{(1-q)}] \geq 0$, with the branch cut along the line $\theta = \theta'$ along the crack surfaces. One can also check that G^i and G^e have the same asymptotic behavior as the crack boundary is approached, i.e. $r \to a$. This is because in that limit plane strain conditions should prevail.

It is interesting to observe that $\sinh s/aN = \mathrm{sgn}\,(y)\,\mathrm{Re}\,[\{r^2 + (|y| + ia)^2\}^{-1/2}]$ so that one can also write G_0^e as

$$G_0^e = A \log (|y| + ia + \sqrt{(r^2 + (|y| + ia)^2)})$$

where the branch $\mathrm{Re}\,[\sqrt{(r^2 + (|y| + ia)^2)}] > 0$ has been taken.

It is worth pointing out that Bueckner (1987) derived the potential G^i for internal circular cracks by assuming $G^i = G^i(s, q)$ and solved the corresponding Laplace equation directly. One can alternatively derive the non-axisymmetric part of the potential $G_s^e = G^e - G_0^e$ in an analogous way. Denoting $\tilde{q} = 1/q = (a/r) \exp i(\theta' - \theta)$ and assuming $G_s^e(x, y, z) = G_s^e(t, \tilde{q})$ ($\tilde{q} \to 0$ corresponds to $q \to \infty$), it can be shown that

$$a^2 N \nabla^2 G_s^e = G_{s,tt}^e - \tan t (G_{s,t}^e - 2\tilde{q} G_{s,\tilde{q}}^e) = 0. \tag{20}$$

Of particular interest are the solutions in the form of

$$G_n^e(x, y, z) = \tilde{q}^n H_n(t) \tag{21}$$

for positive n, where by eqn (20) H_n satisfies

$$H_n''(t) + (2n - 1) \tan t H_n'(t) = 0. \tag{22}$$

One solves eqn (22) and gets $H_n'(t) = B \cos^{(2n-1)} t$ for constant B. It can be shown by eqns (19) that the following relation is valid when constant B is equated to A:

$$G^e_{s,t} = 2\sum_{n=1}^{\infty} G^e_{n,t}; \quad \text{hence} \quad G^e = G^e_0 + 2\sum_{n=1}^{\infty} G^e_n \tag{23}$$

where

$$G^e_n = -A\bar{q}^n \int_t^{\pi/2} \cos^{2n-1} t \,\mathrm{d}t \tag{24}$$

for $n = 1, 2, \ldots$, and where the function G^e_0 is defined in eqns (19). It is also helpful to observe from eqn (24) that $|G^e_n| < A/\cosh^n s \sim O(\zeta^{-n})$ as $\zeta = \sqrt{(r^2+y^2)} \to \infty$. Therefore, G^e_n has the correct asymptotic behavior at the remote field. For an internal circular crack, analogous results were derived by Bueckner (1987) applying the reciprocity theorem to elastic fields being "fundamental" (with $\mathbf{h}_1$ as its displacements) and "regular" (with real displacements due to a point load) on the region bounded by crack surfaces excluding a small cylindrical tube along the crack front. In that case

$$G^i = G^i_0 + 2\sum_{n=1}^{\infty} G^i_n \tag{25}$$

where

$$G^i_n = -Aq^n \int_s^{\infty} (1/\cosh^{2n+1} s)\,\mathrm{d}s \tag{26}$$

for $n = 0, 1, 2, \ldots$. Hence, one has found that the two cases of internal and external circular cracks are in parallel analogy to each other except that G^e_0 depends on both variables s and t as is seen from eqn $(19)_1$.

In the above we have derived Papkovitch–Neuber potentials that represent the tensile mode weight function fields for internal and external circular cracks. These solutions reproduce the internal circular crack results derived earlier by Bueckner (1987) and display some unique feature for the potential G^e of external circular cracks. As discussed in Bueckner's work the tensile mode potential also uniquely determines the shear mode weight function fields. Hence one can also follow Bueckner's construction to obtain the shear mode weight functions for external circular cracks. Before doing so, the shear mode fields for internal circular cracks will first be examined.

SHEAR MODE WEIGHT FUNCTIONS FOR INTERNAL CIRCULAR CRACKS

In contrast to eqns (9), Bueckner has shown that the shear mode weight functions, corresponding to displacements of some shear mode fundamental field, can be described by shearing potentials g_γ, h_γ and ψ_γ as

$$h_{\gamma r} = -2(1-\nu)g_\gamma + y\psi_{\gamma,r}; \quad h_{\gamma y} = -(1-2\nu)\psi_\gamma + y\psi_{\gamma,y}; \quad h_{\gamma\theta} = -2(1-\nu)h_\gamma + y\psi_{\gamma,\theta}/r \tag{27}$$

where γ ranges over 2 and 3 representing in-plane and anti-plane shearing modes, respectively. In writing eqns (27) we have expressed all quantities in the cylindrical coordinate system r, θ, y. One should note that the original derivations in Bueckner (1987) were expressed in Cartesian coordinates. It was also pointed out by Bueckner that these shearing potentials g_γ, h_γ, $-\psi_\gamma$ can be considered as the components of a vector potential the divergence of which vanishes. Following Bueckner (1987), complex potential functions that are also analytic in the complex variable $r\,\mathrm{e}^{i\theta}$ in the crack domain (e.g. $t = 0$ for external circular cracks) are referred to as being "crack-analytic". Bueckner showed that a crack-analytic potential G^* can be used to construct displacements for the following two kinds of shear mode fundamental fields.

First kind

$$u'_r = -2(1-\nu)\,e^{i\theta}\,G^*_{,y} + y\psi'_{,r}, \quad u'_y = -2(1-\nu)\psi' + (y\psi')_{,y}$$
$$u'_\theta = -2i(1-\nu)\,e^{i\theta}\,G^*_{,y} + y\psi'_{,\theta}/r$$

with

$$\psi' = e^{i\theta}\,(G^*_{,r} + iG^*_{,\theta}/r). \tag{28}$$

Second kind

$$u''_r = -2(1-\nu)[(1-\nu)L^*_{,r} + iL^*_{,\theta}/r] + y\psi''_{,r}, \quad u''_y = -2(1-\nu)\psi'' + (y\psi'')_{,y},$$
$$u''_\theta = -2(1-\nu)[-iL^*_{,r} + (1-\nu)L^*_{,\theta}/r] + y\psi''_{,\theta}/r \tag{29}$$

where $\psi'' = -(1-\nu)L^*_{,y}$, and where another potential function L^* is defined as

$$aL^* = (y^2 + a^2 - r^2)G^*_{,y} + 2ryG^*_{,r} + yG^*. \tag{30}$$

It can be shown that the function L^* is harmonic while $L^*_{,y}$ is crack-analytic. One denotes the fundamental field of the first kind generated by G^* through eqns (28) by $\mathbf{u}'(G^*)$ and the fundamental field of the second kind by $\mathbf{u}''(G^*)$. The above two kinds of shear mode fields equilibrate zero body forces and surface forces and give the pure mode 2 and mode 3 shear fundamental fields, i.e. the shear mode weight functions, through linear combinations. The function L^* can be analogously expanded as eqns (23) and (25) so that

$$L^* = L^*_0 + 2\sum_{n=1}^{\infty} L^*_n. \tag{31}$$

The axisymmetric potential L^*_0 represents the part independent of the angular variable θ.

There exists a general principle among the weight functions for cracks of various crack geometries. That is, these weight functions must, in the vicinity of the crack front, behave asymptotically the same as those for a half-plane crack. In other words, the plane strain conditions should prevail when the crack tip is approached. Near the crack boundary $r = a$, asymptotic analysis can be carried out on the behaviors of the weight function fields of the first and second kinds generated by G^i_n ($n = 0, 1, \ldots$), i.e. $\mathbf{u}'(G^i_n)$ and $\mathbf{u}''(G^i_n)$. Comparing $\mathbf{u}'(G^i_n)$ and $\mathbf{u}''(G^i_n)$ with the plane strain elastic fields, one can easily extract the pure mode II and mode III fields $\mathbf{h}_2(G^i_n)$ and $\mathbf{h}_3(G^i_n)$ by linearly combining them in a proper way. Because all these fields are constructed by linear operators to the potential G^i_n and the total weight function fields are summed up according to eqn (25) as

$$\mathbf{h}_j = \mathbf{h}_j(G^i_0) + 2\sum_{n=1}^{\infty} \mathbf{h}_j(G^i_n). \tag{32}$$

In this manner Bueckner (1987) showed that the mode 2 and mode 3 weight function fields are constructed as

$$(2-\nu)\mathbf{h}_2 = \mathbf{u}''(G^i_0)/(1-\nu) + \mathbf{u}''(G^i) - e^{-i\theta'}\,\mathbf{u}'(G^i + G^i_0)$$
$$(2-\nu)\mathbf{h}_3 = -i(1-\nu)\mathbf{u}''(G^i_0) - i\mathbf{u}''(G^i) - i(1-\nu)\,e^{-i\theta'}\,\mathbf{u}'(G^i + G^i_0). \tag{33}$$

Therefore, the shearing potentials for internal circular cracks can be expressed as follows.

Mode II

$$(2-\nu)g_2 = -e^{i(\theta-\theta')}\,(G^i+G^i_0)_{,y}+L^i_{0,r}+(1-\nu)L^i_{,r}+iL^i_{,\theta}/r$$
$$(2-\nu)h_2 = -i\,e^{i(\theta-\theta')}\,(G^i+G^i_0)_{,y}+iL^i_{0,r}-iL^i_{,r}+(1-\nu)L^i_{,\theta}/r$$
$$(2-\nu)\psi_2 = -e^{i(\theta-\theta')}\,(G^i_{,r}+G^i_{0,r}+iG^i_{,\theta}/r)-L^i_{0,y}-(1-\nu)L^i_{,y}. \tag{34}$$

Mode III

$$(2-\nu)g_3 = -i(1-\nu)\,e^{i(\theta-\theta')}\,(G^i+G^i_0)_{,y}+i(1-\nu)(L^i_0-L^i)_{,r}+L^i_{,\theta}/r$$
$$(2-\nu)h_3 = (1-\nu)\,e^{i(\theta-\theta')}\,(G^i+G^i_0)_{,y}-(1-\nu)L^i_{0,r}-L^i_{,r}-i(1-\nu)L^i_{,\theta}/r$$
$$(2-\nu)\psi_3 = -i(1-\nu)[e^{i(\theta-\theta')}\,(G^i_{,r}+G^i_{0,r}+iG^i_{,\theta}/r)+(L^i_0-L^i)_{,y}] \tag{35}$$

where L^i is related to G^i by eqn (30) and L^i_0 is the axisymmetric part of potential L^i. One can show that both G^i_0 and $L^i_{0,y}$ are constant on the crack faces and hence are crack-analytic. It is also important to note that L^i_0 generates the axisymmetric shear mode weight function fields through the fundamental field of the second kind listed in eqns (29).

For convenience we also present here the following quantities that appear in the above eqns (34) and (35):

$$G^i = A\left\{\frac{1}{\sqrt{(q-1)}}\log\frac{\sinh s-\sqrt{(q-1)}}{\sinh s+\sqrt{(q-1)}}+\pi/2-\arctan(\sinh s)\right\}$$
$$G^i_0 = A[\arctan(\sinh s)-\pi/2]$$
$$L^i = \frac{A\sin t(1+q)+(y/a)(qG^i_0+G^i)}{1-q}$$
$$L^i_0 = A\sin t+(y/a)G^i_0. \tag{36}$$

The above results enable one to calculate the shear mode weight functions for internal circular cracks. It is clear from the above results that the shear mode weight functions are determined solely by the tensile mode potential G^i. This method by Bueckner of constructing shear mode solutions from the tensile potential is very important for other crack geometries too, as will be shown in the next section for external circular cracks.

SHEAR MODE WEIGHT FUNCTIONS FOR EXTERNAL CIRCULAR CRACKS

Bueckner's (1987) method of deriving shear mode weight functions from the tensile mode potential for internal circular cracks is extended to the case of external circular cracks. Similarly a potential L^e can be related to G^e by eqn (30). One can note that the axisymmetric potentials G^e_0, $L^e_{0,y}$ are both harmonic but not crack-analytic. Hence Bueckner's fundamental fields of the first and second kinds in eqns (28) and (29) cannot be applied to G^e_0 and L^e_0. The detailed process of deriving the shearing potentials for external circular cracks is similar to Bueckner's derivations for internal circular cracks except that the axisymmetric shear mode weight function field for an external circular crack is generated by the potential $G^e_1 = -A(1-\sin t)\exp(\theta'-\theta)/r$, as defined in eqn (21) for $n = 1$, through the fundamental field of the first kind in eqns (28), in contrast to that generated by G^i_0 through the field of the second kind for internal cracks. A potential L^e_s is defined in terms of G^e_s as

$$aL^e_s = (y^2+a^2-r^2)G^e_{s,y}+2ryG^e_{s,r}+yG^e_s. \tag{37}$$

Analogous asymptotic analysis on $\mathbf{u}'(G^e_n)$ and $\mathbf{u}''(G^e_n)$ ($n = 1,2,\ldots$) allows one to extract the pure mode II and mode III weight function fields from proper linear combinations of them. The potentials G^e_n and G^i_n behave asymptotically similar to each other near the crack front (as they should) and one can take full advantage of the intermediate results presented by Bueckner (1987). Observing that similarity one can state the final results without any further derivation:

$$(2-\nu)\mathbf{h}_2 = \mathbf{u}''(G_s^c) - e^{-i\theta'}\,\mathbf{u}'(G^{II}), \quad (2-\nu)\mathbf{h}_3 = -i\mathbf{u}''(G_s^c) - i\,e^{-i\theta'}\,\mathbf{u}'(G^{III}) \tag{38}$$

where one has adopted the notations $G^{II} = G_s^c - \nu G_1^c$ and $G^{III} = (1-\nu)G_s^c + \nu G_1^c$. The shearing potentials are then expressed as follows.

Mode II

$$\begin{aligned}
(2-\nu)g_2 &= -e^{i(\theta-\theta')}\,G^{II}_{,x} + (1-\nu)L^c_{s,r} + iL^c_{s,\theta}/r \\
(2-\nu)h_2 &= -i\,e^{i(\theta-\theta')}\,G^{II}_{,x} - iL^c_{s,r} + (1-\nu)L^c_{s,\theta}/r \\
(2-\nu)\psi_2 &= -e^{i(\theta-\theta')}\,[G^{II}_{,r} + iG^{II}_{,\theta}/r] - (1-\nu)L^c_{s,x}.
\end{aligned} \tag{39}$$

Mode III

$$\begin{aligned}
(2-\nu)g_3 &= -i\,e^{i(\theta-\theta')}\,G^{III}_{,x} - i(1-\nu)L^c_{s,r} + L^c_{s,\theta}/r \\
(2-\nu)h_3 &= e^{i(\theta-\theta')}\,G^{III}_{,x} - L^c_{s,r} - i(1-\nu)L^c_{s,\theta}/r \\
(2-\nu)\psi_3 &= -i\,e^{i(\theta-\theta')}\,(G^{III}_{,r} + iG^{III}_{,\theta}/r) + i(1-\nu)L^c_{s,x}.
\end{aligned} \tag{40}$$

Therefore, only the function G_s^c, i.e. the non-axisymmetric part of the potential G^c, is needed to construct the shearing potentials g_i, h_i and ψ_i. Listed here are the quantities that appeared in eqns (39) and (40) for the convenience of readers.

$$\begin{aligned}
G_s^c &= \frac{A}{\sqrt{(1-q)}}\log\frac{(\sin t - \sqrt{(1-q)})\,(1+\sqrt{(1-q)})}{(\sin t + \sqrt{(1-q)})\,(1-\sqrt{(1-q)})}, \\
G_1^c &= A(a/r)(\sin t - 1)\exp i(\theta' - \theta) \\
L_s^c &= \frac{2A\sinh s(1-\sin t) + (y/a)G_s^c}{1-q}.
\end{aligned} \tag{41}$$

It should be emphasized that eqns (39)–(41) permit one to calculate the shear mode weight functions for external circular cracks when remote displacements are fully constrained. These results are checked against some special known results in Appendix A.

As discussed previously, in order to compute the weight functions under relaxed remote displacement boundary conditions one needs to superpose equal and oppositely sensed net reaction forces and/or moments at infinity to take off the corresponding displacement restraints.

NEUBER FIELDS

Neuber (1937) solved the elasticity problem of hyperbolic circumferential notches subjected to forces and/or moments at infinity. He derived the full field solutions for the displacements and stresses. An external circular crack can be treated as a degenerated hyperbolic notch. Therefore, the notch solutions given by Neuber can be directly used for external circular cracks. Neuber fields are presented below for later application.

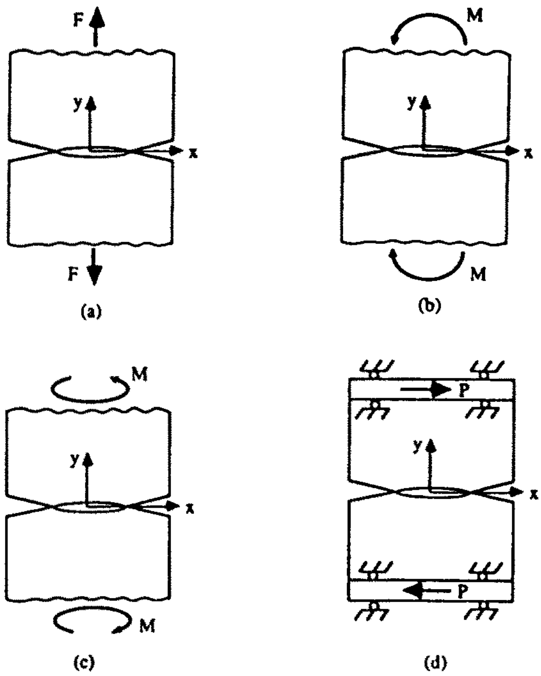

Fig. 2. The loadings of an external circular crack in Neuber fields: (a) tension; (b) bending; (c) torsion; (d) shear.

Remote tension

This case corresponds to an external circular crack subjected to a pair of remote centered forces of magnitude F (Fig. 2(a)). The associated remote displacement $c\ [= u_y(\infty)]$ is

$$c = (1-\nu^2)F/(2Ea). \tag{42}$$

The complete displacement field is expressed as

$$u_r = -(1-2\nu)G^N_{,r} - yG^N_{,ry}, \quad u_\theta = -(1-2\nu)G^N_{,\theta}/r - yG^N_{,y\theta}/r,$$
$$u_y = 2(1-\nu)G^N_{,y} - yG^N_{,yy} \tag{43}$$

where the Neuber potential G^N can be presented in spheroidal coordinates s, t as

$$G^N = \frac{ac}{\pi(1-\nu)}\left(\frac{y}{a}\ \text{arctan}\ (\sinh s) + \sin t - \log\left[\cosh s(1+\sin t)\right]\right). \tag{44}$$

Equations (43) and (44) enable one to determine the displacements throughout the entire cracked body under the action of remote centered forces.

Remote bending

Without loss of generality, consider the crack system subjected to a couple of remote bending moment M about the z-axis (Fig. 2(b)). In this case the induced remote rotation $\vartheta\ [= \partial u_y(\infty)/\partial x]$ is

$$\vartheta = \frac{3M(1-\nu^2)}{4Ea^3}. \tag{45}$$

The full displacement field is similarly expressed by eqns (43) in terms of the following potential G^N:

$$G^N = \frac{a^2\vartheta \cos\theta}{\pi(1-\nu)}\left\{\frac{r}{a}\left[\frac{y}{a}\arctan(\sinh s)+\sin t\right]-\frac{a}{3r}(1-\sin^3 t)-\frac{a}{r}(\sin t-1)\right\}. \quad (46)$$

Torsional field

The displacements due to a couple of remote torque of magnitude M (Fig. 2(c)) constitute a pure torsional field. The remote rotation ϑ [$= u_\theta(\infty)/r$] is

$$\vartheta = \frac{3M(1+\nu)}{8Ea^3}. \quad (47)$$

The displacements for the above torsional field are all zero except

$$u_\theta = \frac{2\vartheta r}{\pi}(\arctan(\sinh s)+\sinh s/\cosh^2 s). \quad (48)$$

Remote shear forces

Consider, without loss of generality, a pair of shear forces of magnitude P in the $\pm x$-directions acting at $y = \pm\infty$ (Fig. 2(d)). The body is restrained against rotation in the x–y plane. In this case the remote displacement c [$= u_x(\infty)$] is

$$c = \frac{(1+\nu)(2-\nu)P}{4aE}. \quad (49)$$

The displacement field is described by three potentials g^N, h^N, ψ^N so that

$$u_r = -2(1-\nu)g^N + y\psi^N_{,r}, \quad u_\theta = -2(1-\nu)h^N + y\psi^N_{,\theta}/r, \quad u_y = -(1-2\nu)\psi^N + y\psi^N_{,y}. \quad (50)$$

The potentials are written as

$$g^N = \frac{c}{\pi(1-\nu)}\left[-\arctan(\sinh s)+\frac{\nu}{2-\nu}\frac{a^2}{r^2}\sinh s(1-\sin t)^2\right]\cos\theta$$

$$h^N = \frac{c}{\pi(1-\nu)}\left[\arctan(\sinh s)+\frac{\nu}{2-\nu}\frac{a^2}{r^2}\sinh s(1-\sin t)^2\right]\sin\theta$$

$$\psi^N = \frac{2c}{\pi(2-\nu)}\frac{a}{r}(1-\sin t)\cos\theta. \quad (51)$$

By the formulae under the remote loading in the above four cases, one can further construct the displacements at an arbitrary spatial position due to arbitrary remote forces and moments.

EFFECT OF REMOTE BOUNDARY CONDITIONS

The weight functions, as displacements of Bueckner's fundamental elastic field, should satisfy the correct displacement boundary condition for the given crack geometry. The weight functions for external circular cracks presented in previous sections are correct only if the pre-assumed zero displacement conditions at infinity are indeed valid. In most of the cases it may not be so because point forces can generate rigid body displacements throughout the entire body. This dependence of the elastic field on the remote displacement boundary conditions was observed by Bueckner (1973). Later Stallybrass (1981) calculated the remote reaction force and moments associated with a wedge opening point force pair on the crack faces and gave the stress intensity factor induced by that force pair under a completely free boundary condition at infinity. Here the effect of the remote boundary conditions is investigated on general weight functions for external circular cracks based on the previously given Neuber solutions.

An external circular crack system is first considered with no restraints at infinity against displacements. Alternatively denote the respective vertical, radial and tangential directions y, r', θ' associated with an angular position θ' along the crack front by $\alpha = 1, 2, 3$, respectively. One may observe that the reciprocal relation in eqn (8) can be extended to shear mode weight function fields so that

$$h^{f}_{\alpha j}(r, \theta, y; \theta', a) = \lim_{r' \to a} \frac{A_{\alpha\beta}}{8} \sqrt{\left(\frac{2\pi}{r'-a}\right)} u^{f}_{\beta j}(r, \theta, y; r', \theta') \tag{52}$$

(superscript f emphasizes the completely free conditions) where the matrix $A_{\alpha\beta}$ is diagonal with

$$A_{11} = A_{22} = E/(1-\nu^2), \quad A_{33} = E/(1+\nu). \tag{53}$$

The matrix $A_{\alpha\beta}$ is proportional to a prelogarithmic energy factor matrix in the expression for the self energy of a straight dislocation line. The notation $u^{f}_{\beta j}(r, \theta, y; r', \theta')$ denotes the jth component of the displacement field at r, θ, y due to a pair of point forces at the crack face location r', θ', pointing in the $\pm\beta$-directions. A wedge opening, radial and tangential point force pair on the crack faces correspond to $\beta = 1, 2, 3$, respectively. The total displacement field $u^{f}_{\alpha j}(r, \theta, y; r', \theta')$ can be divided into two parts

$$u^{f}_{\alpha j} = u_{\alpha j} + \hat{u}_{\alpha j} \tag{54}$$

where the $u_{\alpha j}$ are the displacements under the fully constrained condition at infinity and $\hat{u}_{\alpha j}$ are the additional displacements when the remote restraints are taken off. In fact $\hat{u}_{\alpha j}$ correspond to the Neuber fields generated by the remote reaction forces and moments.

For convenience one can adopt here several notations to denote the Neuber fields associated with remote forces and moments. Let $\xi^{k}_{j}(\mathbf{r})$ and $\eta^{k}_{j}(\mathbf{r})$ denote the jth component of the displacement field at $\mathbf{r}$ associated respectively with a pair of forces and moments of unit magnitude in the k-direction at $\pm\infty$. We understand the direction k to be fixed in space. Subscripts $\alpha = 1, 2, 3$, are used to denote the components of the displacement vector in the y-, r', θ'-directions (i.e. the vertical, radial and tangential directions at r', θ', y). Hence one lets $\Delta\xi^{k}_{\alpha}(r', \theta')$ and $\Delta\eta^{k}_{\alpha}(r', \theta')$ denote the component in the α-direction of crack face displacement discontinuities at location r', θ'. Functions ξ^{k}_{j}, η^{k}_{j}, $\Delta\xi^{k}_{\alpha}(r', \theta')$ and $\Delta\eta^{k}_{\alpha}(r', \theta')$ are readily extracted from the Neuber solutions given in the last section.

Therefore one can write $\hat{u}_{\alpha j}$ as

$$\hat{u}_{\alpha j}(r, \theta, y; r', \theta') = -P_{\alpha k}(r', \theta')\xi^{k}_{j}(r, \theta, y) - M_{\alpha k}(r', \theta')\eta^{k}_{j}(r, \theta, y) \tag{55}$$

where $P_{\alpha k}(r', \theta')$, $M_{\alpha k}(r', \theta')$ are the components in the k-direction of remote reaction force and moment vectors due to a pair of point forces in the $\pm\alpha$-directions at crack face location r', θ'. Applying the reciprocity theorem to the elastic field with unit point force pairs with no displacements at infinity and Neuber fields with remote displacements c and ϑ one finds

$$2cP_{\alpha k}(r', \theta') = -\Delta\xi^{k}_{\alpha}(r', \theta') \quad \text{similarly} \quad 2\vartheta M_{\alpha k}(r', \theta') = -\Delta\eta^{k}_{\alpha}(r', \theta'). \tag{56}$$

Hence the reaction forces and moments at infinity are related to the crack face displacement discontinuities of the Neuber fields.

It is now straightforward to write the remote reaction force $P_{\alpha k}$ and moment $M_{\alpha k}$ associated with a wedge opening ($\alpha = 1$), radial ($\alpha = 2$) and tangential ($\alpha = 3$) unit point force pair at the crack face location r', θ'. The non-zero components of $P_{\alpha k}$ and $M_{\alpha k}$ are

$$P_{1r}(r') = -\frac{2}{\pi}\cos^{-1}\left(\frac{a}{r'}\right)$$

$$P_{2r}(r') = -\frac{2}{\pi}\left[\cos^{-1}\left(\frac{a}{r'}\right) - \frac{\nu}{2-\nu}\frac{a}{r'}\left(1-\frac{a^2}{r'^2}\right)^{1/2}\right]$$

$$P_{3\theta}(r') = -\frac{2}{\pi}\left[\cos^{-1}\left(\frac{a}{r'}\right) + \frac{\nu}{2-\nu}\frac{a}{r'}\left(1-\frac{a^2}{r'^2}\right)^{1/2}\right]$$

$$M_{1\theta}(r') = M_{3r}(r') = -\frac{2r'}{\pi}\left[\cos^{-1}\left(\frac{a}{r'}\right) + \frac{a}{r'}\left(1-\frac{a^2}{r'^2}\right)^{1/2}\right]. \quad (57)$$

The results due to wedge point forces ($\alpha = 1$), P_{1r} and $M_{1\theta}$ match those given by Stallybrass (1981). Stallybrass obtained his results for remote reaction force and moment associated with a wedge opening point force pair acting on the crack faces by integrating the stress fields within the connection area to calculate the net force and moment on each horizontal plane y = const. Here one has shown that Neuber fields greatly simplified the calculations. Equations (57) also suggest that it is most convenient to resolve the remote reaction forces and moments in the r'-, θ'-, y-directions, i.e. $k = r', \theta', y$. The Neuber fields ξ_i^k and η_i^k generated by these forces will depend on the orientation angle θ' so that one may now write them as $\xi_i^k(r, \theta, y; \theta'; a)$ and $\eta_i^k(r, \theta, y; \theta'; a)$. Substituting eqns (54), (55) and (57) into eqn (52), one can show that the weight functions under the completely free condition are given by

$$h^{\mathrm{f}}_{\alpha i}(r, \theta, y; \theta'; a) = h_{\alpha i}(r, \theta, y; \theta'; a)$$
$$+ (A_{\alpha\beta}/\sqrt{(\pi a)})[\mathscr{P}_{\beta k}\xi_i^k(r, \theta, y; \theta'; a) + a.\mathscr{M}_{\beta k}\eta_i^k(r, \theta, y; \theta'; a)] \quad (58)$$

where $\mathscr{P}_{\alpha k}$ and $\mathscr{M}_{\alpha k}$ are constant matrices the non-zero components of which are

$$\mathscr{P}_{1r} = 1/2, \quad \mathscr{P}_{2r} = (1-\nu)/(2-\nu), \quad \mathscr{P}_{3\theta} = 1/(2-\nu), \quad \mathscr{M}_{1\theta} = \mathscr{M}_{3r} = 1. \quad (59)$$

Hence we have completed the derivation of the weight functions under the completely free condition. In eqn (58) the terms containing $\mathscr{P}_{\alpha k}$ and $\mathscr{M}_{\alpha k}$ represent the contributions from $P_{\alpha k}$ and $M_{\alpha k}$.

It has been shown that the difference between the weight function fields under the completely restrained condition and those under the completely free condition is linearly proportional to Neuber fields. In fact, it was observed by Bueckner (private communication) and also suggested by eqns (44) and (46) that the potential

$$G^{\mathrm{f}} = G^{\mathrm{f}}_0 + 2\sum_{n=1}^{\infty} G^{\mathrm{f}}_n \quad (60)$$

where

$$G^{\mathrm{f}}_0 = A[(y/a)\arctan(\sinh s) + \sin t]$$
$$G^{\mathrm{f}}_1 = (3A/2)\{(r/a)[(y/a)\arctan(\sinh s) + \sin t] - (a/3r)\sin t\cos^2 t)\} e^{i(\theta-\theta')}$$
$$G^{\mathrm{f}}_n = G^{\mathrm{c}}_n \quad \text{for} \quad n > 1 \quad (61)$$

directly determines the tensile mode weight function field through eqns (9) under the completely free condition, i.e. when the solid is free to move at infinity.

A solid with an external circular crack has six degrees of freedom for all possible rigid body displacements at infinity, i.e. three translations in the x-, y-, z-directions and three rotations about the x-, y-, z-axes. In the above one has shown that when the remote rigid body displacements are completely suppressed, a pair of point forces at crack face location

r', θ' would be balanced by a reaction force vector $\mathbf{P}_\alpha$ and a moment vector $\mathbf{M}_\alpha$ at infinity. Each of the six components $P_{\alpha j}$, $M_{\alpha j}$ ($j = x, y, z$) is associated with a degree of freedom. Therefore, if some degrees of freedom are lifted to relax the displacement restraints, the associated components of vector reaction force and/or moment should be taken off by superposing equal and oppositely sensed forces and/or moments, respectively. For general mixed mode loading one could have as many as $2^6 = 64$ kinds of different displacement restraints at infinity, ranging from the completely restrained condition to the completely free condition. Nevertheless the derivation of the weight functions for all these different boundary conditions are essentially similar to that presented in this section for the completely free condition except that in those cases only contributions from relevant components of reaction force and moment should be subtracted from $h_{\alpha j}$, the weight functions for the fully restrained condition.

AXISYMMETRIC WEIGHT FUNCTIONS FOR EXTERNAL CIRCULAR CRACKS

In practice one often encounters the simpler case that the applied forces on a cracked body are axisymmetrically distributed. It is desirable to have explicit expressions for the axisymmetric weight functions, defined as

$$\bar{h}_{\alpha j} = \frac{1}{2\pi}\int_0^{2\pi} h_{\alpha j}\, \mathrm{d}\theta \tag{62}$$

for $j = r, \theta, y$. These axisymmetric weight functions are associated with the intensity factors due to axisymmetric ring loads at arbitrary location r, y.

Using the notation

$$\bar{F} = (1/2\pi)\int_0^{2\pi} F\, \mathrm{d}\theta$$

it follows by eqns (23) that $\bar{G}^c_\gamma$ vanishes. From the definition in eqn (37) one observes that $\bar{L}^c_\gamma$ also vanishes. Using Cauchy's integration theorem one can verify the following results for the axisymmetric potentials $\bar{G}^c, \bar{g}_\gamma, \bar{h}_\gamma, \bar{\psi}_\gamma$ as:

$$\bar{G}^c = G^c_0,\quad \bar{g}_2 = -A(a/r)(\sin t)_{,y},\quad \bar{\psi}_2 = A \sin t/aN,\quad \bar{h}_3 = A(a/r)(\sin t)_{,y} \tag{63}$$

(other potentials are zero). These potentials determine the axisymmetric weight functions through

$$\bar{h}_{1r} = -(1-2\nu)G^c_{0,r} - yG^c_{0,ry};\quad \bar{h}_{1y} = 2(1-\nu)G^c_{0,y} - yG^c_{0,yy} \tag{64}$$

$$\bar{h}_{\gamma r} = -2(1-\nu)\bar{g}_\gamma + y\bar{\psi}_{\gamma,r};\quad \bar{h}_{\gamma y} = -(1-2\nu)\bar{\psi}_\gamma + y\bar{\psi}_{\gamma,y};\quad \bar{h}_{\gamma\theta} = -2(1-\nu)\bar{h}_\gamma \tag{65}$$

(other weight function components are zero). An interesting observation is that $\bar{\psi}_2 = 0$ when $t = 0$, and hence $\bar{h}_{2y} = 0$ on the crack faces. That is, the axisymmetric normal loads on the crack faces do not induce a mode 2 singular stress field at the crack tip. This observation generalizes an earlier 2D result by Erdogan (1962).

Under the completely free condition at infinity, additional contributions to the weight functions from the remote restraints need to be superposed. Integrating eqns (58) over the variable θ gives the solutions immediately. For example, for crack face loading one has

$$\bar{h}^{f}_{1y} = \bar{h}_{1y} + \frac{1}{2(\pi a)^{3/2}} \cos^{-1}\left(\frac{a}{r}\right)$$

$$\bar{h}^{f}_{3\theta} = \bar{h}_{3\theta} + \frac{3}{4a(\pi a)^{3/2}} \left[\cos^{-1}\left(\frac{a}{r}\right) + \frac{a^2}{r^2}\sqrt{\left(1-\frac{a^2}{r^2}\right)}\right]. \tag{66}$$

Hence we have provided the axisymmetric weight functions for external circular cracks. Some of these results, e.g. the solutions corresponding to rings of shear loads are believed to be new. It is clear that the remote displacement boundary conditions play an important role in these weight function fields.

CRACK FACE WEIGHT FUNCTIONS FOR EXTERNAL CIRCULAR CRACKS UNDER RESTRAINED DISPLACEMENT BOUNDARY CONDITIONS

In practice, crack face weight functions are often needed in closed form. They play an important role in pursuing the perturbation analysis of calculating the first-order variations of elastic field of a cracked solid when the shape of the crack is perturbed from some reference geometry (Rice, 1985a; Gao and Rice, 1986, 1987a, b; Gao, 1988).

Under the restrained displacement boundary condition at infinity when all rigid body displacements at infinity are fully suppressed, the full-field weight function solutions for external circular cracks are presented in eqns (9) and (19) for the tensile mode and in eqns (27) and (39)–(41) for shear modes. One can carry out the algebra in those expressions on the crack faces where one of the spheroidal coordinates t vanishes. When this is done one finds

$$k_{1y}(r,\theta;\theta';a) = k = \frac{\sqrt{((r^2-a^2)/a\pi^3)}}{d^2}$$

$$k_{2r}(r,\theta;\theta';a) = -\frac{k}{2-\nu}\left\{2\frac{d}{r}\cos\lambda + \frac{a}{r}[\nu-2-2\nu\cos 2\lambda]\right\}$$

$$k_{3r}(r,\theta;\theta';a) = \frac{2k}{2-\nu}\left\{\frac{d}{r}\sin\lambda - \nu\frac{a}{r}\sin 2\lambda\right\}$$

$$k_{2\theta}(r,\theta;\theta';a) = -\frac{2k}{2-\nu}\left\{(1-\nu)\frac{d}{r}\sin\lambda + \nu\frac{a}{r}\sin 2\lambda\right\}$$

$$k_{3\theta}(r,\theta;\theta';a) = \frac{k}{2-\nu}\left\{-2(1-\nu)\frac{d}{r}\cos\lambda + \frac{a}{r}[2-\nu-2\nu\cos 2\lambda]\right\}$$

$$k_{2y}(r,\theta;\theta';a) = k_{3y} = k_{1\theta} = k_{1r} = 0 \tag{67}$$

where d is the distance between r, θ and a, θ', and λ is depicted in Fig. 3. The following supplementary geometrical relations are listed here:

$$\cos\lambda = [a - r\cos(\theta'-\theta)]/d; \quad \sin\lambda = -r\sin(\theta'-\theta)/d$$

$$\cos 2\lambda = 1 - 2r^2\sin^2(\theta'-\theta)/d^2; \quad \sin 2\lambda = -2[a-r\cos(\theta'-\theta)]r\sin(\theta'-\theta)/d^2. \tag{68}$$

The mode I crack face weight functions k_{1j} exist in the literature (Kassir and Sih, 1975) (note that the displacement boundary condition at infinity was not specified in their work).

As verification eqns (67) can be checked against special solutions in two limiting situations: $\nu \to 0$ and $a \to \infty$. In the first case, basic analytical solutions are known and the second case corresponds to a half-plane crack. It is shown in Appendix B that these solutions do approach the correct limits in the above special cases. Besides these limiting cases, eqns

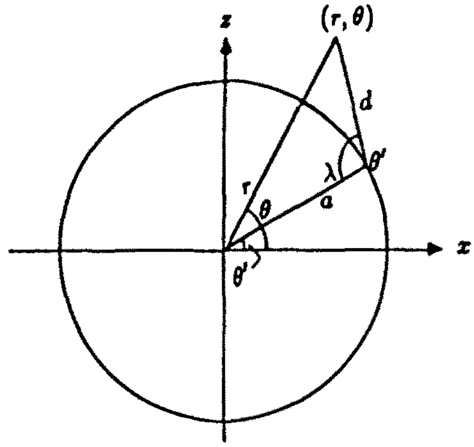

Fig. 3. Geometrical parameters d, λ of an external circular crack.

(67) have been used (work in progress), via Rice's (1985a) perturbation technique, to derive the first-order results for external cracks with fronts deviating slightly from circles. Those results also match the first-order expansion of some of the existing analytic solutions in the literature (e.g. elliptic cracks under shear) for arbitrary values of ν and a.

CRACK FACE WEIGHT FUNCTIONS UNDER OTHER REMOTE BOUNDARY CONDITIONS

There are different forms of weight functions for external circular cracks associated with various displacement boundary conditions at infinity. One can likewise categorize the crack face weight functions according to the remote boundary conditions.

For the completely free condition, i.e. no displacement restraints at infinity, it follows from eqn (58) that

$$k^{f}_{\alpha j}(r,\theta;\theta';a) = k_{\alpha j}(r,\theta;\theta';a) + \frac{A_{\alpha\beta}}{\sqrt{(\pi a)}}[\mathscr{P}_{\beta k}\Delta\xi^{k}_{j}(r,\theta;\theta';a) + a\mathscr{M}_{\beta k}\Delta\eta^{k}_{j}(r,\theta;\theta';a)] \quad (69)$$

where matrices $\mathscr{P}_{\alpha k}$ and $\mathscr{M}_{\alpha k}$ have been listed in eqns (59). The functions $\Delta\xi^{k}_{j}(r,\theta;\theta';a)$ and $\Delta\eta^{k}_{j}(r,\theta;\theta';a)$ denote the crack face displacement discontinuities in Neuber fields.

If the remote boundary condition is such that the remote rigid body displacements are only partially suppressed, then eqn (69) needs to be modified according to the superposition rule discussed above. Distinguish the following contributions to the weight functions from remote reaction forces and moments:

$$\begin{aligned}
&k^{P}_{1\nu} = P_{1\nu}/(2a\sqrt{(\pi a)}), \quad k^{M}_{1\nu} = 3M_{1\theta}\cos(\theta'-\theta)/(2a^{2}\sqrt{(\pi a)}),\\
&k^{M}_{3\theta} = 3M_{3\nu}(r)/(4a^{2}\sqrt{(\pi a)}), \quad k^{P}_{2r} = P_{2r}(r)\cos(\theta-\theta')/(2a\sqrt{(\pi a)}),\\
&k^{P}_{3r} = P_{2r}(r)\sin(\theta-\theta')/(2a\sqrt{(\pi a)}), \quad k^{P}_{2\theta} = P_{3\theta}(r)\sin(\theta-\theta')/(2a\sqrt{(\pi a)}),\\
&k^{P}_{3\theta} = P_{3\theta}(r)\cos(\theta-\theta')/(2a\sqrt{(\pi a)})
\end{aligned} \quad (70)$$

where $P_{\alpha j}(r)$ and $M_{\alpha j}$ have been given in eqns (57).

The following six kinds of typical conditions are considered.

Case (i)

Completely restrained condition, i.e. all displacements vanish at infinity. In this case one simply has

$$k_{\alpha j}^{(i)} = k_{\alpha j} \tag{71}$$

where $k_{\alpha j}$ are expressed in eqns (67).

Case (ii)

Remote displacements all suppressed except for translation in the y-direction. In this case the crack face weight functions are

$$k_{1v}^{(ii)} = k_{1v} + k_{1v}^{P} \tag{72}$$

(all other $k_{\alpha j}^{(ii)} = k_{\alpha j}$). Equation (72) coincides with the solution given by Tada *et al.* (1973), although they did not specify the condition under which their solution would be valid.

Case (iii)

Remote displacements all suppressed except for rotations about the x- and z-axes. In this case

$$k_{1v}^{(iii)} = k_{1v} + k_{1v}^{M} \tag{73}$$

(all other $k_{\alpha j}^{(iii)} = k_{\alpha j}$).

Case (iv)

Remote displacements all suppressed except for translation in the x-direction. In this case the tensile mode crack face weight functions remain unchanged whereas the shear mode results are

$$k_{\gamma j}^{(iv)} = k_{\gamma j} + k_{\gamma j}^{P} \cos \theta \tag{74}$$

where $\gamma = 2, 3$ for shear modes.

Case (v)

Remote displacements all suppressed except for rotation about the y-axes. In this case all other $k_{\alpha j}$ remain unchanged except

$$k_{3\theta}^{(v)} = k_{3\theta} + k_{3\theta}^{M}. \tag{75}$$

Case (vi)

Completely free condition, i.e. no displacement restraints so that all displacements are allowed. In this case

$$k_{\alpha j}^{(vi)} = k_{\alpha j} + k_{\alpha j}^{M} + k_{\alpha j}^{P} \tag{76}$$

for all modes $\alpha = 1, 2, 3$, where contributions from all reaction forces and moments are taken off. The mode 1 result rederives that given by Stallybrass (1981).

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REFERENCES

Bueckner, H. F. (1970). A novel principle for the computation of stress intensity factors. *Z. Angew. Math. Mech.* **50**, 529–546.

Bueckner, H. F. (1973). Field singularities and related integral representations. In *Mechanics of Fracture 1: Methods of Analysis and Solution of Crack Problems* (Edited by G. C. Sih), pp. 239–314. Noordhoff, Leyden, The Netherlands.
Bueckner, H. F. (1977). The weight functions of mode I of the penny-shaped and of the elliptic crack. In *Fracture Mechanics and Technology* (Edited by G. C. Sih and C. L. Chow), Vol. II, pp. 1069–1107. Sijthoff and Noordhoff, Leyden, The Netherlands.
Bueckner, H. F. (1987). Weight functions and fundamental fields for the penny-shaped and the half-plane crack in three space. *Int. J. Solids Structures* **23**, 57–93.
Erdogan, F. (1962). On the stress distribution on plates with colinear cuts under arbitrary loads. Proceedings, Fourth U.S. National Congress of Applied Mechanics, p. 547.
Galin, L. A. (1953). *Contact Problems in the Theory of Elasticity* (in Russian) (Translated by H. Moss and edited by I. N. Sneddon). School of Physical Sciences and Applied Mathematics, North Carolina State College Publications (1961).
Gao, H. (1988). Nearly circular shear mode cracks. *Int. J. Solids Structures* **24**, 177–193.
Gao, H. and Rice, J. R. (1986). Shear stress intensity factors for a planar crack with slightly curved front. *J. Appl. Mech.* **53**, 774–778.
Gao, H. and Rice, J. R. (1987a). Somewhat circular tensile cracks. *Int. J. Fracture* **33**, 155–174.
Gao, H. and Rice, J. R. (1987b). Nearly circular connections of elastic half spaces. *J. Appl. Mech.* **54**, 627–634.
Love, A. E. (1927). *Treatise on the Mathematical Theory of Elasticity*. Cambridge University Press, London.
Kassir, M. K. and Sih, G. (1975). *Three Dimensional Crack Problems*. Noordhoff, Leyden, The Netherlands.
Neuber, H. (1937). *Kerpspennungslehre*. Springer, Berlin (1937 and 1958), English translation available from Edward Bros., Ann Arbor, Michigan.
Paris, P. C., McMeeking, R. M. and Tada, H. (1976). The weight function method for determining stress intensity factors. In *Cracks and Fracture*, STP 601, pp. 471–489. ASTM, Philadelphia.
Rice, J. R. (1972). Some remarks on elastic crack tip stress fields. *Int. J. Solids Structures* **8**, 751–758.
Rice, J. R. (1985a). First order variations in elastic fields due to variation in location of a planar crack front. *J. Appl. Mech.* **52**, 571–579.
Rice, J. R. (1985b). Three dimensional elastic crack tip interactions with transformation strains and dislocations. *Int. J. Solids Structures* **21**, 781–791.
Sih, G. C. and Liebowitz, H. (1968). Mathematical theories of brittle fracture. In *Fracture* (Edited by H. Liebowitz), Vol. II, pp. 67–190. Academic Press, New York.
Stallybrass, M. P. (1981). On the concentrated loading of certain elastic half-space problems and related external crack problems. A new approach. *Int. J. Engng Sci.* **21**, 781–791.
Tada, H., Paris, P. C. and Irwin, G. R. (1973). *The Stress Analysis of Cracks Handbook*. Del Research Corporation, Hellertown, Pennsylvania.

APPENDIX A: WEIGHT FUNCTION SOLUTIONS IN A SPECIAL CASE

The weight function solutions for an external circular crack presented here are novel. It is hence desired to show its consistency with some existing special case solutions. For example, the stress intensity factors induced along the front of an external circular crack by concentrated forces on the y-axis were given by Kassir and Sih (1975). These results give the special weight function solutions for external circular cracks on the y-axis ($t = \pi/2$). Through some algebraic manipulation one obtains from eqns (27) and (39)–(41) of the text the weight functions on the y-axis as

$$h_{1x} = \frac{Aa\cos(\theta'-\theta)}{a^2+y^2}\left(1-2\nu-\frac{2y^2}{a^2+y^2}\right)$$

$$h_{1y} = \frac{Ay}{a^2+y^2}\left(1-2\nu+\frac{2y^2}{a^2+y^2}\right)$$

$$h_{2x} = \frac{(1-\nu)Ay\cos(\theta'-\theta)}{(2-\nu)(a^2+y^2)}\left(3-2\nu+\frac{2(2-\nu)a^2}{(1-\nu)(a^2+y^2)}\right)$$

$$h_{2y} = -\frac{Aa}{a^2+y^2}\left(1-2\nu+\frac{2y^2}{a^2+y^2}\right)$$

$$h_{3x} = -\frac{(1-\nu)(3-2\nu)Ay\sin(\theta'-\theta)}{(2-\nu)(a^2+y^2)} \tag{A1}$$

($h_{3y} = 0$). The solutions of eqns (A1) match the results presented by Kassir and Sih (1975). Kassir and Sih failed to specify the proper boundary condition needed for the above results to be valid. That is, eqns (A1) are correct only when remote displacements are fully constrained, i.e. no displacements at infinity.

Again one can easily find the solutions under the completely free condition by eqns (58) of the text. For example, it is trivial to show from eqns (58) that

$$h^{f}_{1y} = \frac{1}{2(\pi a)^{1/2}}\left[\arctan(y/a)+\frac{ay}{a^2+y^2}\left\{1-\frac{a^2}{(1-\nu)(a^2+y^2)}\right\}\right]. \tag{A2}$$

This is consistent with a solution for the mode I stress intensity factor due to a pair of tensile forces acting on the y-axis given in Tada *et al.* (1973), although they did not specify the proper remote boundary condition for the above solution to be valid.

By definition the above weight function results can be used to construct solutions for the intensity factors induced by an arbitrary distribution of forces along the y-axis under remote displacement restraint conditions. A point force in the tangential θ-direction on the y-axis is irrelevant here since it is equivalent to a radial force in the $\theta+\pi/2$ polar direction.

APPENDIX B: CHECKS OF THE CRACK FACE WEIGHT FUNCTION FORMULAE

The crack face weight functions presented in eqn (67) can be checked against the following special limiting cases:

Case 1. $\nu = 0$

In the absence of body forces three harmonic functions F_x, F_z, H (Sih and Liebowitz, 1968) can be used to generate the special field of shear mode cracks. While the following is satisfied:

$$H_{,y} = F_{x,x} + F_{z,z} \tag{B1}$$

the displacement field is represented as

$$\begin{aligned} u_x &= -2[(1-\nu^2)/E]F_x + [(1+\nu)/E]yH_{,x} \\ u_y &= -[(1+\nu)/E][(1-2\nu)H + yH_{,y}] \\ u_z &= -2[(1-\nu^2)/E]F_z + [(1+\nu)/E]yH_{,z}. \end{aligned} \tag{B2}$$

The stress field is derived by stress–strain relations as

$$\begin{aligned} \sigma_{yy} &= yH_{,yy}; \quad \sigma_{yz} = -(1-\nu)F_{z,y} + \nu H_{,z} + yH_{,zy} \\ \sigma_{yx} &= -(1-\nu)F_{x,y} + \nu H_{,x} + yH_{,xy}. \end{aligned} \tag{B3}$$

Now without loss of generality, consider that the crack faces are loaded with a pair of unit concentrated forces in the $\pm x$-direction acting at the location x', z'. The elasticity problems of a shear mode crack this loaded are completely governed by the harmonic functions F_x, F_z and H, hence can be formulated as seeking these unknown potentials that vanish at infinity having the boundary condition that on the plane $y = 0$

$$F_x|_{y=0^+} = F_z|_{y=0^+} = 0, \quad \text{outside crack area} \tag{B4}$$

and

$$\left.\begin{aligned} &[-(1-\nu)F_{z,y} + \nu H_{,z}]_{y=0} = 0, \\ &[-(1-\nu)F_{x,y} + \nu H_{,x}]_{y=0} = \delta(x-x')\delta(z-z'), \end{aligned}\right\} \quad \text{within crack area.} \tag{B5}$$

Although the solutions to the harmonic function F_x, F_z, H under the above boundary condition is difficult to find, the special case of $\nu = 0$ turns out to be interesting. In that case it is easy to show that $F_z = 0$ and F_x satisfies $\nabla^2 F_x = 0$ with

$$\begin{aligned} F_{x,y} &= -\delta(x-x')\delta(z-z'), \quad \text{within crack area} \\ F_x &= 0, \quad \text{outside crack area.} \end{aligned} \tag{B6}$$

Comparing the above with the eqns (3) of the text, one can note immediately that F_x should have the same mathematical solutions as the tensile mode potential function Y. It also follows from eqns (B3) and eqns (2) of the text that on the crack plane $y = 0$, σ_{yx} has exactly the same solution as the tensile mode stress distribution σ_{yy}. Therefore, one can reach the conclusion that in the case of $\nu = 0$ the shear mode cracks have the same distribution of the stress field as that of tensile cracks, and the traction on the plane $y = 0$ is in the same direction as that of the concentrated force pair. Therefore, it is elementary to show from the definition of the stress intensity factor that when $\nu = 0$

$$\begin{aligned} k_{2r} &= k\cos(\theta'-\theta); \quad k_{2\theta} = -k\sin(\theta'-\theta) \\ k_{3r} &= k\sin(\theta'-\theta); \quad k_{3\theta} = k\cos(\theta'-\theta). \end{aligned} \tag{B7}$$

Now letting $\nu = 0$ in eqns (67) of the text, and using the geometrical relations of eqns (68), one can see that those solutions match eqns (B7).

Case 2: $a \to \infty$, *i.e. a half-plane crack*

One may observe that in the limit $a \to \infty$, the circular crack becomes a half-plane crack. Therefore eqns (67), in the same limit, should approach the corresponding solutions for crack face weight functions for a half-plane crack. Assuming in that limit the crack front lies along the z-axis and $x > 0$ denotes the crack face (Fig. B1), the polar coordinates r, θ in the crack plane are replaced by Cartesian coordinates x, z in the following manner:

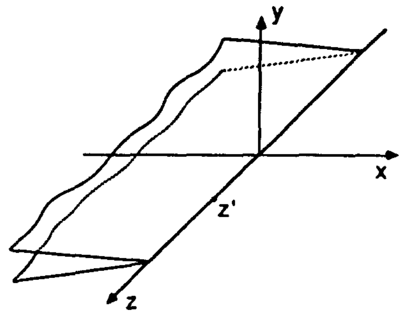

Fig. B1. A half-plane crack lying on the $y = 0$ plane with its tip along the z-axis.

$$r-a \to x; \quad a\theta \to -z. \tag{B8}$$

Using the following asymptotic relations when $a \to \infty$

$$\begin{aligned} &d \sin \lambda = -z'+z; \quad d \cos \lambda = -x \\ &d^2 \sin 2\lambda = 2x(z'-z); \quad d^2 \cos 2\lambda = x^2-(z'-z)^2 \end{aligned} \tag{B9}$$

where now $d^2 = x^2+(z'-z)^2$ is the square of the distance between a point x, z on the crack face and a point $0, z'$ along the crack front. It can be shown that eqns (67) are reduced to

$$\begin{aligned} k_{1y} &= k = \frac{(2x/\pi^3)^{1/2}}{[x^2+(z'-z)^2]}, \quad k_{1x} = k_{1z} = k_{2y} = k_{3y} = 0 \\ k_{2r} &= k_{2x} = \left[1+\frac{2\nu}{2-\nu}\frac{x^2-(z'-z)^2}{x^2+(z'-z)^2}\right]k \\ k_{3\theta} &= k_{3z} = \left[1-\frac{2\nu}{2-\nu}\frac{x^2-(z'-z)^2}{x^2+(z'-z)^2}\right]k \\ -k_{2\theta} &= k_{2z} = \frac{4\nu}{2-\nu}\frac{x(z'-z)}{x^2+(z'-z)^2}k \\ -k_{3r} &= k_{3x} = \frac{4\nu}{2-\nu}\frac{x(z'-z)}{x^2+(z'-z)^2}k. \end{aligned} \tag{B10}$$

Equations (B10) match the correct point force intensity factor formulae for a half-plane crack (Tada *et al.*, 1973).